

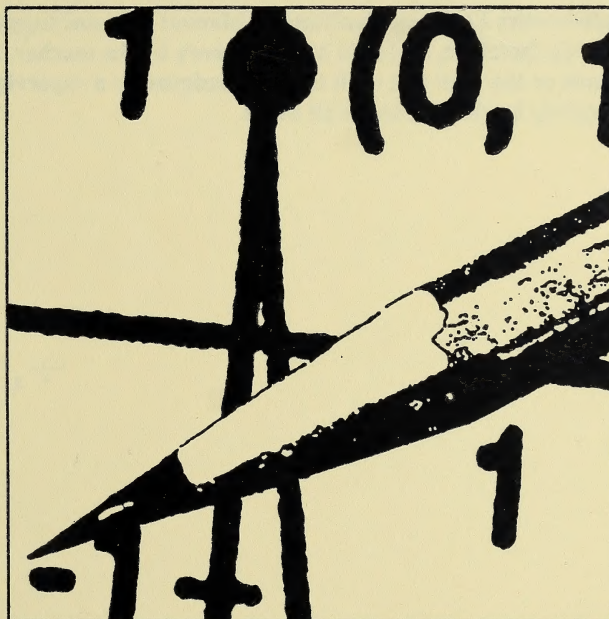


MATHEMATICS 3

LEARNING FACILITATOR'S MANUAL

CANADIAN

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UNIT 2: DIFFERENTIATION OF ALGEBRAIC EXPRESSIONS AND GRAPHING



Distance
Learning

Alberta
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Note

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Mathematics 31
Learning Facilitator's Manual
Unit 2
Differentiation of Algebraic Expressions and Graphing
Alberta Distance Learning Centre
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Topic 1: Differentiation of Algebraic Expressions

- ⑤ 1. If $y = \frac{1}{3}x^3 - 5x^2 + 8 - \frac{2}{x^2}$, find $\frac{dy}{dx}$.

$$y = \frac{1}{3}x^3 - 5x^2 + 8 - 2x^{-2}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{3}{3}x^2 - 5(2)x + 0 - 2(-2)x^{-3} \\ &= x^2 - 10x + 4x^{-3}\end{aligned}$$

- ⑥ 2. If $y = 3x^{\frac{3}{4}} - \frac{1}{2}x^{-\frac{1}{2}}$, find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= 3\left(\frac{3}{4}\right)x^{\frac{3}{4}-1} - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} \\ &= \frac{9}{4}x^{-\frac{1}{4}} + \frac{1}{4}x^{-\frac{3}{2}}\end{aligned}$$

- ⑥ 3. If $y = (x^3 - 2x^2 + x - 5)(2x^3 - 8)$, find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= (x^3 - 2x^2 + x - 5)D_x(2x^3 - 8) + (2x^3 - 8)D_x(x^3 - 2x^2 + x - 5) \\ &= (x^3 - 2x^2 + x - 5)(6x^2) + (2x^3 - 8)(3x^2 - 4x + 1) \\ &= 6x^5 - 12x^4 + 6x^3 - 30x^2 + 6x^5 - 8x^4 + 2x^3 - 24x^2 + 32x - 8 \\ &= 12x^5 - 20x^4 + 8x^3 - 54x^2 + 32x - 8\end{aligned}$$

⑧

4. If $y = \frac{5x}{\sqrt{x^2 + 7}}$, find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2 + 7)^{\frac{1}{2}} D_x (5x) - (5x) D_x (x^2 + 7)^{\frac{1}{2}}}{x^2 + 7} \\ &= \frac{(x^2 + 7)^{\frac{1}{2}} (5) - (5x) \left(\frac{1}{2}\right) (x^2 + 7)^{-\frac{1}{2}} (2x)}{x^2 + 7} \\ &= \frac{5(x^2 + 7)^{\frac{1}{2}} - 5x^2 (x^2 + 7)^{-\frac{1}{2}}}{(x^2 + 7)} \\ &= \frac{5(x^2 + 7)^{-\frac{1}{2}} (x^2 + 7 - x^2)}{(x^2 + 7)} \\ &= \frac{35(x^2 + 7)^{-\frac{1}{2}}}{(x^2 + 7)} \\ &= \frac{35}{(x^2 + 7)^{\frac{3}{2}}}\end{aligned}$$

⑤

5. If $y = \frac{7}{x^{\frac{1}{3}}}$, find $\frac{dy}{dx}$.

$$\begin{aligned}y &= 7x^{-\frac{1}{3}} \\ \frac{dy}{dx} &= 7\left(-\frac{1}{3}\right)x^{-\frac{1}{3}-1} \\ &= -\frac{21}{5}x^{-\frac{4}{3}}\end{aligned}$$

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6. If $y = \frac{5x^2 - 3x + 4}{x^3 - 2x^2 + 7}$, find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^3 - 2x^2 + 7)D_x(5x^2 - 3x + 4) - (5x^2 - 3x + 4)D_x(x^3 - 2x^2 + 7)}{(x^3 - 2x^2 + 7)^2} \\&= \frac{(x^3 - 2x^2 + 7)(10x - 3) - (5x^2 - 3x + 4)(3x^2 - 4x)}{(x^3 - 2x^2 + 7)^2} \\&= \frac{10x^4 - 20x^3 + 70x - 3x^3 + 6x^2 - 21 - 15x^4 + 9x^3 - 12x^2 + 20x^3 - 12x^2 + 16x}{(x^3 - 2x^2 + 7)^2} \\&= \frac{-5x^4 + 6x^3 - 18x^2 + 86x - 21}{(x^3 - 2x^2 + 7)^2}\end{aligned}$$

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7. If $y = \sqrt{x^2 - 3}(x^2 - 5)^{-2}$, find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \sqrt{x^2 - 3}D_x(x^2 - 5)^{-2} + (x^2 - 5)^{-2}D_x\sqrt{x^2 - 3} \\&= \sqrt{x^2 - 3}(-2)(x^2 - 5)^{-3}(2x) + (x^2 - 5)^{-2}\left(\frac{1}{2}\right)(x^2 - 3)^{-\frac{1}{2}}(2x) \\&= (x^2 - 3)^{\frac{1}{2}}(x^2 - 5)^{-3}(-4x) + (x^2 - 5)^{-2}(x^2 - 3)^{-\frac{1}{2}}(x) \\&= x(x^2 - 3)^{-\frac{1}{2}}(x^2 - 5)^{-3}[-4(x^2 - 3) + (x^2 - 5)] \\&= x(x^2 - 3)^{-\frac{1}{2}}(x^2 - 5)^{-3}(-4x^2 + 12 + x^2 - 5) \\&= x(x^2 - 3)^{-\frac{1}{2}}(x^2 - 5)^{-3}(-3x^2 + 7)\end{aligned}$$

5

8. If $y = -2(3x^4 - 5)^{-\frac{1}{2}}$, find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= (-2) \left(-\frac{1}{2} \right) (3x^4 - 5)^{-\frac{3}{2}} (12x^3) \\ &= 12x^3 (3x^4 - 5)^{-\frac{3}{2}}\end{aligned}$$

5

9. If $y = (2x - 3)\sqrt{x + 1}$, find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= (2x - 3)D_x (x + 1)^{\frac{1}{2}} + (x + 1)^{\frac{1}{2}} D_x (2x - 3) \\ &= (2x - 3) \frac{1}{2} (x + 1)^{-\frac{1}{2}} + (x + 1)^{\frac{1}{2}} (2) \\ &= \frac{1}{2} (x + 1)^{-\frac{1}{2}} (2x - 3) + 2(x + 1)^{\frac{1}{2}} \\ &= \frac{1}{2} (x + 1)^{-\frac{1}{2}} [(2x - 3) + 4(x + 1)] \\ &= \frac{1}{2} (x + 1)^{-\frac{1}{2}} (2x - 3 + 4x + 4) \\ &= \frac{6x + 1}{2\sqrt{x + 1}}\end{aligned}$$

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10. Find the equation of the tangent to the graph of the relation $x^2 + y^2 = 16$ at $P(4, 0)$.

$$\begin{aligned}x^2 + y^2 &= 16 \\ 2x + 2yy' &= 0 \\ y' &= \frac{-2x}{2y} \\ &= \frac{-x}{y}\end{aligned}$$

At $P(4, 0)$ the slope is $\frac{-4}{0}$.

Topic 1

The slope is undefined; thus, the equation of the tangent is $x = 4$.

_____ marks

Topic 2: Graphing Polynomial Functions

- ⑤ 1. Find the local maximum and local minimum points for the function $f(x) = 2x^3 + 15x^2 + 24x - 7$.

$$f(x) = 2x^3 + 15x^2 + 24x - 7$$

$$f'(x) = 6x^2 + 30x + 24$$

$$= 6(x^2 + 5x + 4)$$

$$= 6(x+1)(x+4)$$

To find the extreme values (turning points), let $f'(x) = 0$.

$$6(x+1)(x+4) = 0$$

$$x = -1 \text{ or } x = -4$$

When $x = -1$, $y = -18$; hence, $P(-1, -18)$ is a turning point.

When $x = -4$, $y = 9$; hence, $Q(-4, 9)$ is a turning point.

$$f''(x) = 12x + 30$$

$P(-2, -18)$ is a minimum point since $f''(-2) > 0$.

$Q(-4, 9)$ is a maximum point since $f''(-4) < 0$.

- ② 2. Which of the following is the second derivative of $f(x) = 6x^2 - 10x - 3x^{-1} + 9$?

A. $12x - 10 + 4x^{-2}$

B. $12x + 10 + 6x$

C. $6(2 - x^{-3})$

D. $12 + 6x^{-3}$

$$y = 6x^2 - 10x - 3x^{-1} + 9$$

$$y' = 12x - 10 + 3x^{-2}$$

$$y'' = 12 - 6x^{-3}$$

$$= 6(2 - x^{-3})$$

Therefore, the answer is C.

②

3. The function $f(x) = 2x^3 + 15x^2 + 24x - 7$ has a point of inflection. Which of the following is this point?

A. $\left(-\frac{9}{2}, -\frac{5}{2}\right)$

B. $\left(\frac{9}{2}, -\frac{5}{2}\right)$

C. $\left(-\frac{5}{2}, -\frac{9}{2}\right)$

D. $\left(\frac{5}{2}, -\frac{9}{2}\right)$

$$y = 2x^3 + 15x^2 + 24x - 7$$

$$y' = 6x^2 + 30x + 24$$

$$y'' = 12x + 30$$

To find a critical value, let $y'' = 0$.

$$12x + 30 = 0$$

$$x = -\frac{5}{2}$$

Therefore, the answer is C.

④

4. Show that the graph of $y = -x^3 - x^2 + 8x + 12$ is tangent to the x -axis; then, determine the coordinates of the tangent point.

$$y = -x^3 - x^2 + 8x + 12$$

$$y' = -3x^2 - 2x + 8$$

When y is a maximum or a minimum, $y' = 0$.

$$-3x^2 - 2x + 8 = 0$$

$$-(3x^2 + 2x - 8) = 0$$

$$-(3x - 4)(x + 2) = 0$$

$$x = \frac{4}{3} \text{ or } x = -2$$

$$\text{At } x = \frac{4}{3}, y = 18\frac{14}{27}.$$

$$\text{At } x = -2, y = 0.$$

The graph touches the x -axis at $x = -2$. Thus, the coordinates of the tangent point are $(-2, 0)$.

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5. If $f(x) = (x+2)^3(1-x)$, find the following:

a. the first and second derivatives

$$\begin{aligned}\frac{dy}{dx} &= (1-x)(3)(x+2)^2 + (x+2)^3(-1) \\ &= (x+2)^2(3-3x-x-2) \\ &= (x+2)^2(1-4x)\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= (1-4x)(2)(x+2) + (x+2)^2(-4) \\ &= (x+2)(2-8x-4x-8) \\ &= (x+2)(-12x-6) \\ &= -6(x+2)(2x+1)\end{aligned}$$

b. the points of inflection

$$\text{When } \frac{d^2y}{dx^2} = 0, \quad -6(x+2)(2x+1) = 0.$$

$$x = -2 \quad \text{or} \quad x = -\frac{1}{2}$$

$$f(-2) = 0$$

For points close to $(-2, 0)$ where $x > -2$ or $x < -2$, $\frac{dy}{dx}$ is positive.

Therefore, $(-2, 0)$ is a point of inflection.

$$\begin{aligned}f\left(-\frac{1}{2}\right) &= \left(-\frac{1}{2} + 2\right)^3 \left(1 - \left(-\frac{1}{2}\right)\right) \\ &= \left(\frac{3}{2}\right)^3 \left(\frac{3}{2}\right) \\ &= \left(\frac{3}{2}\right)^4 \\ &= \frac{81}{16}\end{aligned}$$

For points close to $\left(-\frac{1}{2}, \frac{81}{16}\right)$ where $x > -\frac{1}{2}$ or $x < -\frac{1}{2}$, $\frac{dy}{dx}$ is positive.

Therefore, $\left(-\frac{1}{2}, \frac{81}{16}\right)$ is a point of inflection.

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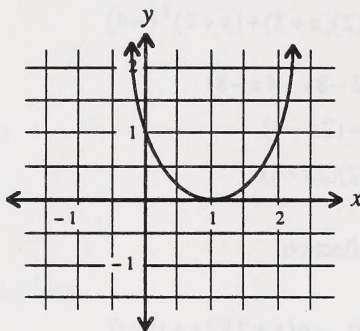
6. Determine whether or not a point of inflection exists at $(1, 0)$ on the curve $y = (x - 1)^4$.

If $y = (x - 1)^4$, then $y' = 4(x - 1)^3$ and $y'' = 12(x - 1)^2$.

At $x = 1$, $y'' = 0$; thus, it is possible that $(1, 0)$ is a point of inflection.

However, for points close to $(1, 0)$, $y' < 0$ in the interval $x < 1$, and $y' > 0$ in the interval $x > 1$.

Therefore, the point $(1, 0)$ is not a point of inflection of the curve defined by $y = (x - 1)^4$.



- ⑥ 7. Show that the function given by $y = x^3 - 2x^2 + 3x$ has neither minima nor maxima for any real value of x .

$$y = x^3 - 2x^2 + 3x$$

$$y' = 3x^2 - 4x + 3$$

$$\text{If } y' = 0, \text{ then } 3x^2 - 4x + 3 = 0.$$

$$\begin{aligned} b^2 - 4ac &= 16 - (4)(3)(3) \\ &= -20 \end{aligned}$$

Since $b^2 - 4ac < 0$, these points do not exist. Therefore, there are no real roots for any local maximum or local minimum points.

- ④ 8. State the local maximum and local minimum values of $f(x) = x^3 - 3x + 4$. Use the second-derivative test to determine which is a local maximum and which is a local minimum.

$$f(x) = x^3 - 3x + 4$$

$$D_x y = 3x^2 - 3$$

$$\text{Let } D_x y = 0.$$

$$3(x^2 - 1) = 0$$

$$x^2 - 1 = 0$$

$$x = \pm 1$$

$$\text{At } x = 1, f(x) = 2.$$

$$\text{At } x = -1, f(x) = 6.$$

$$D_x^2 y = 6x$$

$$\text{At } x = +1, D_x^2 y = 6. \quad (D_x^2 y \text{ is positive; thus, } 2 \text{ is a minimum.})$$

$$\text{At } x = -1, D_x^2 y = -6. \quad (D_x^2 y \text{ is negative; thus, } 6 \text{ is a maximum.})$$

Therefore, 2 is a minimum value and 6 is a maximum value.

- ⑥ 9. Determine the intervals where the curve defined by $f(x) = 2x^3 - 9x^2 - 24x - 12$ is concave up or concave down.

$$y = 2x^3 - 9x^2 - 24x - 12$$

$$y' = 6x^2 - 18x - 24$$

$$y'' = 12x - 18$$

To find a critical value, let $y'' = 0$.

$$12x - 18 = 0$$

$$x = \frac{3}{2}$$

Let $y' = 0$.

$$6(x^2 - 3x - 4) = 0$$

$$6(x - 4)(x + 1) = 0$$

$$x = 4 \text{ or } x = -1$$

At $x = 4$, $y = -124$ and $y'' = 30$; thus, -124 is a minimum.

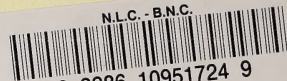
At $x = -1$, $y = 1$ and $y'' = -30$; thus, 1 is a maximum.

Therefore, the curve is concave up when $x > \frac{3}{2}$ and concave down when $x < \frac{3}{2}$.

Topic 2

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